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Notions of independence related to free groups II: from Singleton Condition to Singleton Independence

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1. Introduction. This note is the successive part of Prof. Accardi's lecture in this volume. Motivated by the central limit problem for algebraic probability spaces arising from the Haagerup states on the free group with countably infinite generators, we introduce a new notion of statistical independence in terms of inequalities rather than of usual algebraic identities. In the case of the Haagerup states the role of the Gaussian law is played by the Ullman distribution. The limit process is realized explicitly on the finite temperature Boltzmannian Fock space. Furthermore, a functional central limit theorem associated with the Haagerup states is proved and the limit white noise is investigated.

2. Singleton Condition. A quick review. In order to prove a central limit theorem with the method of moments it is necessary to observe that *only a few* singletons give a non-zero contribution to the limit. The role of the singleton condition was first pointed out by von Waldenfels [28], [29]. The content of this section is rather standard and is included for completeness.

DEFINITION 1. Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{C}$ a C^*-algebra with norm $|\cdot|$, and $E:\mathcal{A}\to\mathcal{C}$ a real linear map. A finite or countably infinite set of sequences

$$(b_n^{(1)})_{n=1}^{\infty},\quad (b_n^{(2)})_{n=1}^{\infty},\cdots,(b_n^{(j)})_{n=1}^{\infty},\cdots$$

of elements in $\mathcal{A}$ with mean $E(b_n^{(j)}) = 0$ is said to satisfy the *singleton condition* with respect to E if for any choice of $k \geq 1$, $j_1, \cdots, j_k \in \mathrm{N}$, and $n_1, \cdots, n_k \in \mathrm{N}$

$$E\big(b_{n_1}^{(j_1)} \cdots b_{n_k}^{(j_k)}\big) = 0 \tag{1}$$

holds whenever there exists an index n_s which is different from all other ones, i.e., such that $n_s \neq n_t$ for $s \neq t$.

In the above definition the condition $E(b_n^{(j)}) = 0$ is, in fact, a consequence of (1). The singleton condition is equivalent to the usual independence in the classical case and follows from free independence [27]. We may generalize the (E, ψ)-independence [10] by replacing the condition $E(b_n^{(j)}) = 0$ with $\psi(b_n^{(j)}) = 0$.

DEFINITION 2. We say that sequences $(b_n^{(1)}), (b_n^{(2)}), \cdots$ of elements of $\mathcal{A}$ satisfy the condition of *boundedness of the mixed momenta* if for each $k \in \mathrm{N}$ there exists a positive constant $\nu_k \geq 0$ such that

$$\left|E\big(b_{n_1}^{(j_1)} \cdots b_{n_k}^{(j_k)}\big)\right| \leq \nu_k \tag{2}$$

for any choice of $n_1, \cdots, n_k$ and $j_1, \cdots, j_k$.

Given a sequence $b = (b_n)_{n=0}^{\infty} \subset \mathcal{A}$, we put

$$S_N(b) = \sum_{n=1}^{N} b_n. \tag{3}$$

LEMMA 1. *Let $(b_n^{(1)}), (b_n^{(2)}), \cdots$ be sequences of elements of $\mathcal{A}$ satisfying the condition of boundedness of the mixed momenta. Then, for any $\alpha > 0$ it holds that*

$$\lim_{N\to\infty} E\left(\frac{S_N(b^{(1)})}{N^\alpha} \cdot \frac{S_N(b^{(2)})}{N^\alpha} \cdots \frac{S_N(b^{(k)})}{N^\alpha}\right)$$

$$= \lim_{N\to\infty} N^{-\alpha k} \sum_{\alpha k \leq p \leq k} \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,p\} \\ \text{surjective}}} \sum_{\substack{\sigma:\{1,\cdots,p\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b_{\sigma\circ\pi(1)}^{(1)} \cdots b_{\sigma\circ\pi(k)}^{(k)}\right), \tag{4}$$

in the sense that one limit exists if and only if the other does and the limits coincide. (The limit is understood in the sense of norm convergence in C.)

LEMMA 2. *Notations and assumptions being the same as in Lemma1, assume that the sequences $(b_n^{(j)})$ satisfies the singleton condition with respect to E. Then*

$$\lim_{N\to\infty} E\left(\frac{S_N(b^{(1)})}{N^\alpha} \cdot \frac{S_N(b^{(2)})}{N^\alpha} \cdots \frac{S_N(b^{(k)})}{N^\alpha}\right) = 0 \tag{5}$$

takes place if $\alpha > 1/2$ or if $\alpha = 1/2$ and k is odd. If $\alpha = 1/2$ and k is even, say $k = 2n$, the left hand side of (5) is equal to the limit

$$\lim_{N\to\infty} N^{-n} \sum_{\substack{\pi:\{1,\cdots,2n\}\to\{1,\cdots,n\} \\ 2-1 \text{ map}}} \sum_{\substack{\sigma:\{1,\cdots,n\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b_{\sigma\circ\pi(1)}^{(1)} \cdots b_{\sigma\circ\pi(2n)}^{(2n)}\right). \tag{6}$$

Moreover, the following Gaussian bound takes place:

$$\limsup_{N\to\infty}\left|E\left(\frac{S_N(b^{(1)})}{N^{1/2}}\cdot\frac{S_N(b^{(2)})}{N^{1/2}}\cdots\frac{S_N(b^{(2n)})}{N^{1/2}}\right)\right|\le\frac{(2n)!}{2^n n!}\nu_{2n}. \qquad (7)$$

3. Properties of the Haagerup States. In the notations of Section 1, the two sequences $\{(g_n), (g_n^{-1})\}$ satisfy the singleton condition with respect to the Haagerup state φ_γ only when $\gamma = 0$. However, φ_γ satisfies a weak analogue of the singleton condition. When the state φ_γ under consideration is fixed, we write for simplicity

$$\tilde{g}_\alpha = g_\alpha - \gamma.$$

Obviously $\varphi_\gamma(\tilde{g}_\alpha) = 0$.

DEFINITION 3. (i) A product $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}$ is called *separable* at k, $1 \le k \le m$, if $\alpha_p \ne \alpha_q^*$ whenever $1 \le p \le k < q \le m$.

(ii) $\tilde{g}_{\alpha_k}$ is called a *singleton* in the product $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}$ if $\tilde{g}_{\alpha_k} \ne \tilde{g}_{\alpha_l}^*$ for any $l \ne k$.

(iii) Let $\tilde{g}_{\alpha_k}$ be a singleton in the product $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}$. It is called *outer* if $\tilde{g}_{\alpha_p} \ne \tilde{g}_{\alpha_q}^*$ for any $p < k < q$.

(iv) A singleton $\tilde{g}_{\alpha_k}$ is called *inner* if $\tilde{g}_{\alpha_p} = \tilde{g}_{\alpha_q}^*$ for some $p < k < q$.

For example, in the product $\tilde{g}_1\tilde{g}_2\tilde{g}_1^{-1}\tilde{g}_3\tilde{g}_2$, the second $\tilde{g}_2$ is an inner singleton and the forth $\tilde{g}_3$ and the last $\tilde{g}_2$ are outer singletons. Notice that $\tilde{g}_2$ is not a "singleton" in the sense that $\tilde{g}_2$ appears twice, cf. Definition 1.

LEMMA 3. *If $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}$ is separable at k, then*

$$\varphi_\gamma(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}) = \varphi_\gamma(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})\varphi_\gamma(\tilde{g}_{\alpha_k+1}\cdots\tilde{g}_{\alpha_m})$$

LEMMA 4. *If $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}$ has an outer singleton, then*

$$\varphi(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}) = 0.$$

Proof. If $\tilde{g}_{\alpha_k}$ is an outer singleton, applying Lemma3 twice we find

$$\begin{aligned}\varphi_\gamma(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}) &= \varphi_\gamma(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})\varphi_\gamma(\tilde{g}_{\alpha_{k+1}}\cdots\tilde{g}_{\alpha_m})\\ &= \varphi_\gamma(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_{k-1}})\varphi_\gamma(\tilde{g}_{\alpha_k})\varphi_\gamma(\tilde{g}_{\alpha_{k+1}}\cdots\tilde{g}_{\alpha_m}) = 0,\end{aligned}$$

as desired. ∎

The next result is a generalization of von Waldenfels' argument [28], [29] to products with inner singletons.

LEMMA 5. *Assume that a product $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_m}$ has no singleton at all or has no outer singletons. Let s be the number of inner singletons in the product and let*

$$p = |\{g_j;\ \textit{there exist } 1 \le k, l \le m \textit{ such that } \alpha_k = (j,+),\ \alpha_l = (j,-)\}|$$

Then

$$s \le m-2 \quad \textit{and} \quad p \le \frac{m-s}{2} \qquad (8)$$

Proof. Since there is no outer singleton, there exist at least two factors $\tilde{g}_{\alpha_k}$ and $\tilde{g}_{\alpha_l}$ with $\alpha_k^* = \alpha_l$. Hence $m \geq 2$ and $s \leq m-2$. If $\tilde{g}_{\alpha_l}$ is not a singleton, there exists at least one element $\tilde{g}_{\alpha_k}$ such as $\alpha_k^* = \alpha_l$ and then $j_k = j_l$ $(k \neq l)$. Therefore $2p + s \leq m$. ■

DEFINITION 4. Assume that a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ contains $s \geq 0$ inner singletons and no outer singletons. Let $\alpha_{j_1}, \cdots, \alpha_{j_s}$ be the suffices which correspond the singletons and denote the rest by $\beta_1, \cdots, \beta_{m-s}$ in order. We say that the product satisfies the condition (NCI) if $g_{\beta_1} \cdots g_{\beta_{m-s}} = e$.

LEMMA 6. *If the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ consists only of non-crossing pair partitions and of s inner singletons then*

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = (-\gamma)^s + (-\gamma)^{s+1} P(\gamma) \tag{9}$$

where P is a polynomial. If the (NCI) condition is not satisfied then

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = (-\gamma)^{s+1} P(\gamma). \tag{10}$$

From Lemma6 one can deduces the central limit theorem for the Haagerup states. For more detailed argument see [4].

THEOREM 7. *Let $NCI_m(s, \varepsilon)$ be the set of equivalence classes of products $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ with the index $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_m)$, which consist of $p = (m-s)/2$ non-crossing pairs and of s inner singletons. Then,*

$$\lim_{N \to \infty} \varphi_{\lambda/\sqrt{N}}(\tilde{a}_N^{\varepsilon_1} \cdots \tilde{a}_N^{\varepsilon_k}) = \sum_{s=0}^{m-2} (-\lambda)^s \cdot |NCI_m(s, \varepsilon)|, \tag{11}$$

where

$$a_N^+ = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_j, \qquad a_N^- = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_j^{-1}. \tag{12}$$

Remark. In the previous paper [4] we proved the existence of the limit and obtained an explicit realization of the GNS space of the limit by means of a *finite temperature* analogue of the usual Boltzmannian Fock space. This finite temperature analogue, which was first introduced by Fagnola [13], appears also in the stochastic limit of quantum electrodynamics at finite temperature [1], [3] and, hence, possesses a similar characteristic as the finite temperature (or universally invariant) Brownian motion. As for the symmetrized random variable $Q_N = a_N^+ + a_N^-$, the limit $\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{Q}_N^k)$ is investigated in [18] for any $k \geq 1$ and $\lambda > 0$, and coincides with the k-th moment of

$$u_\lambda(s)ds = \frac{1}{2\pi} \chi_{[-2-\lambda, 2-\lambda]}(s) \frac{\sqrt{(2+\lambda+s)(2-\lambda-s)}}{1-\lambda s} ds$$

which belongs to the Ullman family of probability measures introduced in connection with potential theory. Beyond potential theory the Ullman distributions also have emerged naturally in quantum probability and in physics, see e.g., [1], [10], [19].

4. Limit Process. By a general theory [2] there exist an algebraic probability space $\{\mathcal{A}_\lambda, \psi_\lambda\}$ and two random variables a_λ, a_λ^+ such that

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{a}_N^{\epsilon_1}\cdots\tilde{a}_N^{\epsilon_k}) = \psi_\lambda(a_\lambda^{\epsilon_1}\cdots a_\lambda^{\epsilon_k}). \tag{13}$$

For $\nu = L, R$ let

$$\Gamma(\mathbf{C})_\nu = \mathbf{C} \oplus \bigoplus_{n=1}^{\infty} \mathbf{C}^{\otimes n} \quad \left(= \bigoplus_{n=0}^{\infty} \mathbf{C} \right)$$

denote two copies of the full Fock spaces over $\mathbf{C}$ with free creations a_ν^+ and free annihilation a_ν. Let $\mathcal{H} = \bigoplus_{m,n=0}^{\infty} \mathcal{H}_{m,n}$ be the free product $\Gamma(\mathbf{C})_L * \Gamma(\mathbf{C})_R$, that is, the (m,n)-particle space $\mathcal{H}_{m,n}$ is the complex linear span of the set of vectors $\{a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi\}$ which satisfy the following conditions:

$$|\{j \mid \nu_j = L\}| = m, \qquad |\{j \mid \nu_j = R\}| = n.$$

and the scalar product is given by

$$\left\langle a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi,\ a_{\nu_1'}^+ \cdots a_{\nu_l'}^+ \Phi \right\rangle_{\mathcal{H}} = \begin{cases} 1, & \text{if } (\nu_1, \cdots, \nu_k) = (\nu_1', \cdots, \nu_l'), \\ 0, & \text{otherwise} \end{cases}$$

The actions of the creation operators

$$L^+ := a_L^+ * 1 : \mathcal{H}_{m,n} \to \mathcal{H}_{m+1,n}\ ; \quad R^+ := 1 * a_R^+ : \mathcal{H}_{m,n} \to \mathcal{H}_{m,n+1}$$

are given respectively by

$$\begin{aligned} L^+ a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi &= a_L^+ a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi \\ R^+ a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi &= a_R^+ a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi \end{aligned}$$

and the action of the annihilation

$$L = a_L * 1 : \mathcal{H}_{m,n} \to \mathcal{H}_{m-1,n}\ ; \quad R = 1 * a_R : \mathcal{H}_{m,n} \to \mathcal{H}_{m,n-1}$$

is given by

$$L a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi = \begin{cases} a_{\nu_2}^+ \cdots a_{\nu_k}^+ \Phi, & \text{if } \nu_1 = L \text{ and } k \geq 2 \\ \Phi, & \text{if } \nu_1 = L \text{ and } k = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$R a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi = \begin{cases} a_{\nu_2}^+ \cdots a_{\nu_k}^+ \Phi, & \text{if } \nu_1 = R \text{ and } k \geq 2 \\ \Phi, & \text{if } \nu_1 = R \text{ and } k = 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $P : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{H}_{0,0}^\perp$. Put

$$A_\lambda^- = L^+ + R - \lambda P, \qquad A_\lambda^+ = L + R^+ - \lambda P,$$

where $\lambda \geq 0$ is a constant.

THEOREM 8. *The limit process $(a_\lambda^+, a_\lambda^-, \psi_\lambda)$ is represented on $\mathcal{H}$. That is, all its correlations (13) are given by*

$$\psi_\lambda(a_\lambda^{\epsilon_1} \cdots a_\lambda^{\epsilon_m}) = \langle \Phi, A_\lambda^{\epsilon_1} \cdots A_\lambda^{\epsilon_m} \Phi \rangle_{\mathcal{H}}.$$

Proof. In Theorem7 we have seen that the ψ_λ-correlators are completely determined by the cardinalities of the sets NCI_m. We thus need only to establish a bijective correspondence between NCI_m-partitions associated with $a_\lambda^{\epsilon_1} \cdots a_\lambda^{\epsilon_m}$ and terms in the expansion of

$$\langle \phi, A_\lambda^{\epsilon_1} \cdots A_\lambda^{\epsilon_m} \phi \rangle = \sum_{B_{\nu_1}^{\epsilon_1}, \ldots, B_{\nu_m}^{\epsilon_m}} \langle \phi, B_{\nu_1}^{\epsilon_1} \cdots B_{\nu_m}^{\epsilon_m} \phi \rangle,$$

where $B_R^- = L^+$, $B_L^- = R$, $B_R^+ = R^+$, $B_L^+ = L$ and $B_0^- = B_0^+ = -\lambda P$. In a product $B_{\nu_1}^{\epsilon_1} \cdots B_{\nu_m}^{\epsilon_m}$, we call $(B_{\nu_p}^{\epsilon_p}, B_{\nu_q}^{\epsilon_q})$ $(p < q)$ a *pair* if $B_{\nu_p}^{\epsilon_p} = L$ and $B_{\nu_q}^{\epsilon_q} = L^+$ or $B_{\nu_p}^{\epsilon_p} = R$ and $B_{\nu_q}^{\epsilon_q} = R^+$. If $B_{\nu_p}^{\epsilon_p} = -\lambda P$ we call it a *singleton*. From the definition of $\mathcal{H}, A_\lambda^+, A_\lambda$ we see easily that $\langle \phi, B_{\nu_1}^{\epsilon_1} \cdots B_{\nu_m}^{\epsilon_m} \phi \rangle \neq 0$ if and only if $B_{\nu_1}^{\epsilon_1} \cdots B_{\nu_m}^{\epsilon_m}$ forms a non-crossing pair partition with s inner singletons $(0 \leq s \leq m-2)$. In this case,

$$\langle \phi, B_{\nu_1}^{\epsilon_1} \cdots B_{\nu_m}^{\epsilon_m} \phi \rangle = (-\lambda)^s.$$

Therefore we obtain the desired bijective correspondence. ∎

5. Functional Central Limit Theorem for the Haagerup State. In general, a central limit theorem is extended in a canonical manner to a functional central limit theorem (or invariance principle) from which the corresponding process is derived, see e.g., [26]. Given a sequence $\{b_i\}$ of random variables, for the functional central limit theorem we consider

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{\lfloor Nt \rfloor} b_i = \frac{1}{\sqrt{N}} \int_0^\infty \sum_{i=1}^{\lfloor Nt \rfloor} \chi_{(i-1,i)}(s) b_i ds,$$

which is in the limit $N \to \infty$ equivalent to

$$\frac{1}{\sqrt{N}} \int_0^\infty \chi_{[0,Nt]}(s) \sum_{i=1}^\infty \chi_{(i-1,i)}(s) b_i ds = \frac{1}{\sqrt{N}} \sum_{i=1}^\infty b_i \int_{i-1}^i \chi_{[0,Nt]}(s) ds$$
$$= \frac{1}{\sqrt{N}} \sum_{i=1}^\infty b_i \int_{i-1}^i \chi_{[0,t]}\left(\frac{s}{N}\right) ds.$$

Thus, we consider more generally

$$\frac{1}{\sqrt{N}} \sum_{i=1}^\infty b_i \int_{i-1}^i f\left(\frac{s}{N}\right) ds,$$

where f is a suitable test function.

Going back to our case, we put

$$S_N^{(\varepsilon)}(f) = \sum_{i=1}^{\infty} \tilde{g}_i^{\varepsilon} \int_{i-1}^{i} f\left(\frac{t}{N}\right) dt, \qquad \varepsilon = \pm 1,$$

where f is an $\mathbf{R}$-valued continuous function with compact support. Then we calculate the mixed momenta:

$$\varphi_\gamma\left(\frac{S_N^{(\epsilon_1)}(f_1)}{\sqrt{N}} \cdots \frac{S_N^{(\epsilon_m)}(f_m)}{\sqrt{N}}\right) =$$

$$= \frac{1}{(\sqrt{N})^m} \sum_{i_1,\cdots,i_m=1}^{\infty} \varphi_\gamma(\tilde{g}_{i_1}^{\epsilon_1} \cdots \tilde{g}_{i_m}^{\epsilon_m}) \int_{i_1-1}^{i_1} f_1\left(\frac{t_1}{N}\right) dt_1 \cdots \int_{i_m-1}^{i_m} f_m\left(\frac{t_m}{N}\right) dt_m, \quad (14)$$

where $\varepsilon_j = \pm 1$ and f_j is a continuous function with compact support, $j = 1, 2, \cdots, m$. In view of the uniform bound $\|f_j\|_{L^1} \leq C$ we apply the arguments in Section 4 (only non-crossing pair partitions with inner singletons contribute to the limit). Then, in the limit (14) is equivalent to

$$\frac{1}{(\sqrt{N})^m} \sum_{s=0}^{m-2} (-\gamma)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)}$$

$$\times \sum_{\substack{i_{\omega(1)},\cdots,i_{\omega(s)} \\ \text{distinct}}} \int_{i_{\omega(1)}-1}^{i_{\omega(1)}} f_{\omega(1)}\left(\frac{t}{N}\right) dt \cdots \int_{i_{\omega(s)}-1}^{i_{\omega(s)}} f_{\omega(s)}\left(\frac{t}{N}\right) dt$$

$$\times \sum_{\substack{i_{\alpha(j)}\notin\{\omega(1),\cdots,\omega(s)\} \\ \text{distinct}}} \prod_{j=1}^{p} \int_{i_{\alpha(j)}-1}^{i_{\alpha(j)}} \int_{i_{\alpha(j)}-1}^{i_{\alpha(j)}} f_{\alpha(j)}\left(\frac{t_{\alpha(j)}}{N}\right) f_{\beta(j)}\left(\frac{t_{\beta(j)}}{N}\right) dt_{\alpha(j)} dt_{\beta(j)}$$

$$+O\left(\frac{1}{\sqrt{N}}\right), \quad (15)$$

where $p = (m-s)/2$ and

$$NCI_m(s,\epsilon) = \left\{ \begin{array}{c} (\alpha,\beta,\omega) = (\alpha(1),\cdots,\alpha(p),\beta(1),\cdots,\beta(p),\omega(1),\cdots,\omega(s)); \\ \{\alpha(1),\cdots,\alpha(p),\beta(1),\cdots,\beta(p),\omega(1),\cdots,\omega(s)\} = \{1,\cdots,m\}, \\ \alpha(j) < \beta(j),\ \alpha(j) < \alpha(j+1),\ \omega(j) < \omega(j+1),\ \varepsilon_{\alpha(j)} = -\varepsilon_{\beta(j)}, \\ \text{for each } l \text{ there exists } j \text{ such that } \alpha(j) < \omega(l) < \beta(j) \end{array} \right\}.$$

In (15), the indices $i_{\alpha(j)}$'s and $i_{\omega(j)}$'s are different each other. But again by the uniform boundedness of f_j's, one obtains, for instance,

$$\sum_{i_{\omega(1)}\notin\{i_{\alpha(1)},\cdots,i_{\alpha(p)},i_{\omega(2)},\cdots,i_{\omega(s)}\}} \int_{i_{\omega(1)}-1}^{i_{\omega(1)}} f_{\omega(1)}\left(\frac{t}{N}\right) dt$$

$$= \int_0^\infty f_{\omega(1)}\left(\frac{t}{N}\right) dt + O\left(\frac{1}{N}\right) = N \int_0^\infty f_{\omega(1)}(s) ds + O\left(\frac{1}{N}\right)$$

and

$$\sum_{i_{\alpha(1)}\notin\{i_{\alpha(2)},\cdots,i_{\alpha(p)},i_{\omega(1)},\cdots,i_{\omega(s)}\}} \int_{i_{\alpha(1)}-1}^{i_{\alpha(1)}} \int_{i_{\alpha(1)}-1}^{i_{\alpha(1)}} f_{\alpha(1)}\left(\frac{t_1}{N}\right) f_{\beta(1)}\left(\frac{t_2}{N}\right) dt_1 dt_2$$
$$= \sum_{i=1}^{\infty} \int_{i-1}^{i} \int_{i-1}^{i} f_{\alpha(1)}\left(\frac{t_1}{N}\right) f_{\beta(1)}\left(\frac{t_2}{N}\right) dt_1 dt_2 + O\left(\frac{1}{N^2}\right)$$
$$= N^2 \sum_{i=1}^{\infty} \int_{(i-1)/N}^{i/N} \int_{(i-1)/N}^{i/N} f_{\alpha(1)}(s_1) f_{\beta(1)}(s_2) ds_1 ds_2 + O\left(\frac{1}{N^2}\right).$$

Recall that $\gamma = O(1/\sqrt{N})$. Then (15) becomes

$$\frac{1}{(\sqrt{N})^m} \sum_{s=0}^{m-2} (-\gamma)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} N\int_0^{\infty} f_{\omega(1)}(s)ds \cdots N\int_0^{\infty} f_{\omega(s)}(s)ds$$
$$\times \sum_{i_{\alpha(1)},\cdots,i_{\alpha(p)}=1}^{\infty} \prod_{j=1}^{p} N^2 \int_{(i_{\alpha(j)}-1)/N}^{i_{\alpha(j)}/N} \int_{(i_{\alpha(j)}-1)/N}^{i_{\alpha(j)}/N} f_{\alpha(j)}(s_{\alpha(j)}) f_{\beta(j)}(s_{\beta(j)}) ds_{\alpha(j)} ds_{\beta(j)}$$
$$+O\left(\frac{1}{\sqrt{N}}\right). \tag{16}$$

LEMMA 9. *Let f_1, f_2 be continuous functions with compact supports. Then,*

$$\lim_{N\to\infty} N \sum_{i=1}^{\infty} \int_{(i-1)/N}^{i/N} \int_{(i-1)/N}^{i/N} f_1(s_1) f_2(s_2) ds_1 ds_2 = \int_0^{\infty} f_1(s) f_2(s) ds.$$

The proof is easy. By this lemma the limit of (16) as $N \to N$ becomes

$$\lim_{N\to N} \frac{1}{(\sqrt{N})^m} \sum_{s=0}^{m-2} (-\gamma)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} N\int_0^{\infty} f_{\omega(1)}(s)ds \cdots N\int_0^{\infty} f_{\omega(s)}(s)ds$$
$$\times N\int_0^{\infty} f_{\alpha(1)}(s) f_{\beta(1)}(s)ds \cdots N\int_0^{\infty} f_{\alpha(p)}(s) f_{\beta(p)}(s)ds$$
$$= \sum_{s=0}^{m-2} (-\lambda)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} \prod_{i=1}^{s} \int_0^{\infty} f_{\omega(i)}(s)ds \prod_{j=1}^{p} \int_0^{\infty} f_{\alpha(j)}(s) f_{\beta(j)}(s)ds.$$

Consequently,

THEOREM 10. *For $j = 1, 2, \cdots, m$ let $f_j : \mathbf{R} \to \mathbf{R}$ be a continuous function with compact support. Then one has*

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}} \left(\frac{S_N^{(\epsilon_1)}(f_1)}{\sqrt{N}} \cdots \frac{S_N^{(\epsilon_m)}(f_m)}{\sqrt{N}} \right)$$

$$= \sum_{s=0}^{m-2}(-\lambda)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} \prod_{i=1}^{s}\int_0^\infty f_{\omega(i)}(s)ds \prod_{j=1}^{p}\int_0^\infty f_{\alpha(j)}(s)f_{\beta(j)}(s)ds.$$

The above is a functional central limit theorem. We now put $S_{N,t}^{(\epsilon)}(f) = S_N^{(\epsilon)}(\chi_{[0,t]}f)$. By modifying the above argument, we obtain

THEOREM 11. *For continuous functions f_j, $j = 1, 2, \cdots, m$, with compact supports, we have*

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}\left(\frac{S_{N,t_1}^{(\epsilon_1)}(f_1)}{\sqrt{N}}\cdots\frac{S_{N,t_m}^{(\epsilon_m)}(f_m)}{\sqrt{N}}\right)$$

$$= \sum_{s=0}^{m-2}(-\lambda)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} \prod_{i=1}^{s}\langle 1, f_{\omega(i)}\rangle_{t_{\omega(i)}} \prod_{j=1}^{(m-s)/2} \langle f_{\alpha(j)}, f_{\beta(j)}\rangle_{\min\{t_{\alpha(j)},t_{\beta(j)}\}}$$

where

$$\langle f, g\rangle_t = \int_0^t f(s)g(s)ds.$$

Now we have the Fock representation of this process. Let $\mathcal{H}$ be the Fock space introduced in Section 6, and $\mathcal{K} = L^2(\mathbf{C})$. Using the notations in Section 5, put

$$A_{\lambda,t}^{-}(f) = L^{+}\otimes\chi_{[0,t]}f + R\otimes\chi_{[0,t]}f - \lambda\langle 1, f\rangle_t P,$$

$$A_{\lambda,t}^{+}(f) = L\otimes\chi_{[0,t]}f + R^{+}\otimes\chi_{[0,t]}f - \lambda\langle 1, f\rangle_t P.$$

Then by Theorem8 and Theorem11, we have

THEOREM 12. *The limit process $(a_t^+, a_t^-, \psi_\lambda)$ is represented on $\mathcal{H}\otimes\mathcal{K}$, and its all correlators are given by*

$$\psi_\lambda(a_{t_1}^{\epsilon_1}(f_1)\cdots a_{t_m}^{\epsilon_m}(f_m)) = \left\langle \Phi, A_{\lambda,t_1}^{\epsilon_1}(f_1)\cdots A_{\lambda,t_m}^{\epsilon_m}(f_m)\Phi\right\rangle_{\mathcal{H}\otimes\mathcal{K}}.$$

6. Singleton Independence. We are led to the following

DEFINITION 5. Let $\mathcal{A}$ be a $*$-algebra and let $S = \{g_j, g_j^*\,;\, j\in\mathbf{N}\}$ be a countable subset of $\mathcal{A}$. Assume we are given a family of states φ_γ, $\gamma \geq 0$, on $\mathcal{A}$ such that $\varphi_\gamma(g_\alpha) = \gamma$ for any g_α, where $\alpha = (j,\epsilon)$ and $g_\alpha = g_j^\epsilon$. Then the sequence $\{g_j\}$ is called to be *singleton independent* with respect to φ_γ if

$$|\varphi_\gamma(g_{\alpha_1}\cdots g_{\alpha_k})| \leq \gamma c_k |\varphi(g_{\alpha_1}\cdots \hat{g}_{\alpha_s}\cdots g_{\alpha_k})|, \tag{17}$$

whenever α_s is a singleton for $(\alpha_1, \cdots, \alpha_k)$.

The case of $\gamma = 0$ is reduced to the usual singleton condition. Condition (17) and boundedness (2) implies that

$$|\varphi_\gamma(g_{\alpha_1}\cdots g_{\alpha_m})| \leq C_m\gamma^s \tag{18}$$

whenever $g_{\alpha_1}\cdots g_{\alpha_k}$ has s singletons

Conditions (17), (18) are easily verified for the Haagerup states. Another examples are found in the unitary representations of the free groups [14]. By specializing a parameter of spherical functions associated with representations of the principal series, we obtain a family of positive definite functions:

$$\psi_N(x)=\left(1+|x|\frac{N-1}{N}\right)(2N-1)^{-|x|/2},\qquad x\in F_N,$$

where F_N is the free group on N generators. This state satisfies the singleton independence. In fact, one sees that

$$\psi_N=\left(1+\frac{N-1}{N}\gamma\frac{\partial}{\partial\gamma}\right)\varphi_\gamma,$$

where φ_γ is a Haagerup state with $\gamma=1/\sqrt{2N-1}$. Suppose that $g=g_{\alpha_1}\cdots g_{\alpha_k}$ has s singltons. Then $\varphi_\gamma(g)=\gamma^t$ with some $t\geq s$ and $\psi_N(g)=\gamma^s P(\gamma)$ where P is a polynomial. Since $\psi_N(g_j)=a=\sqrt{2N-1}/N\geq\gamma$, the singleton independence $|\psi_N(g)|\leq C_k a^s$ holds.

As before, we put

$$S_N^{(\varepsilon)}=\sum_{j=1}^N\tilde{g}_j^\varepsilon,\qquad \varepsilon=\pm1,$$

and, for fixed $k\in\mathrm{N}$ and $\varepsilon_1,\cdots,\varepsilon_k\in\{\pm1\}$ we consider the product

$$S_N^{(\varepsilon_1)}\cdots S_N^{(\varepsilon_k)}=\sum_{j_1,\cdots,j_k=1}^N\tilde{g}_{j_1}^{\varepsilon_1}\cdots\tilde{g}_{j_k}^{\varepsilon_k}=\sum_{j_1,\cdots,j_k}\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}.$$

Put $I_k=\{(1,\varepsilon_1),\cdots,(k,\varepsilon_k)\}$ and consider α as a function $\alpha: I_k\to\{1,\cdots,N\}$. For given α put $p=|\alpha(I_k)|$. We denote by $\alpha(I_k)=\{\overline{\alpha}_1,\cdots,\overline{\alpha}_p\}$ its range (with $\overline{\alpha}_i\neq\overline{\alpha}_j$) and put

$$S_j=\alpha^{-1}(\overline{\alpha}_j),\qquad j=1,\cdots,p,$$

$$\mathcal{P}_{k,p}=\{(S_1,\cdots,S_p)\,;\ \text{partition of } I_k \text{ of cardinality } p\},$$

$$[S_1,\cdots,S_p]=\{\alpha\,;\ \alpha|_{S_j}=\alpha(S_j)=\ \text{const. and } \alpha(S_i)\neq\alpha(S_j) \text{ if } i\neq j\}.$$

With these notations our goal is to study the large N asymptotics of the rescaled expectation values

$$\varphi_{\lambda/\sqrt{N}}\left(\frac{S_N^{(\varepsilon_1)}}{\sqrt{N}}\cdots\frac{S_N^{(\varepsilon_k)}}{\sqrt{N}}\right)=N^{-k/2}\sum_{p=1}^k\sum_{(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}}\sum_{\alpha\in[S_1,\cdots,S_p]}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}).\tag{19}$$

LEMMA 13. *Given $s=0,1,\cdots,k$, denote*

$$\mathcal{P}_{k,p}^s=\{(S_1,\cdots,S_p)\ \textit{which have exactly } s \textit{ singletons}\},$$

where a singleton of $(S_1,\cdots,S_p)$ *stands for* S_i *with* $|S_i| = 1$. *Then it holds that* $p \le (k+s)/2$. *Moreover, if* $p < (k+s)/2$ *then*

$$\lim_{N\to\infty} N^{-k/2} \sum_{(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}} \sum_{\alpha\in[S_1,\cdots,S_p]} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}) = 0.$$

Proof. For $(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}$ we have

$$k = \sum_{j=1}^{p}|S_j| = \sum_{\{j\in\{1,\cdots,p\},|S_j|\ge 2\}} |S_j| + s \ge 2(p-s)+s = 2p-s.$$

Then, in view of the boundedness of the mixed momenta (2), we see that the sum is dominated by a constant times of

$$N^{-(k+s)/2}|\mathcal{P}^s_{k,p}|\frac{\lambda^s}{p!}N^p \to 0.$$

■

We see from Lemma13 that the non trivial contribution to the limit of (19) comes from those partitions $(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}$ satisfying $p=(k+s)/2$, that is, $k=2p-s$.

LEMMA 14. *Assume that* $k = 2p-s$ *holds. Then for any* $(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}$, *it holds that* $|S_j| = 1$ *or* $|S_j|=2$ *for all* j.

Proof. Suppose otherwise, say, $|S_1|\ge 3$. Then we have

$$\begin{aligned} k &= 3+\sum_{j\ge 2,|S_j|\ge 2}|S_j| + s \ge 3+2(p-s-1)+s \\ &= 3+2p-2s-2+s = 2p-s+1, \end{aligned}$$

which is incompatible with $k=2p-s$. ■

Suppose that a partition $(S_1,\cdots,S_p)$ of $\{1,\cdots,k\}$ has s singletons and $|S_j| = 1$ or 2 for $j=1,\cdots,p$. We denote by $(\tilde{S}_1,\cdots,\tilde{S}_{p-s})$ the set of all S_j's with $|S_j|=2$ and say that $(\tilde{S}_1,\cdots,\tilde{S}_{p-s})$ is the pair partition associated to $(S_1,\cdots,S_p)$. The pair partition associated to a 2–1 map $\beta:\{1,\cdots,2p\}\to\{1,\cdots,p\}$ shall be called *negligible* if

$$|\varphi_\gamma(g_{\beta_1}\cdots g_{\beta_{2p}})| \le c\gamma. \tag{20}$$

LEMMA 15. *Suppose that* φ_γ *satisfies condition (20). Fix* $s=0,\cdots,k$ *and let* $\tilde{\mathcal{P}}_{k,1,2,s}$ *denote the set of all partitions* $(S_1,\cdots,S_p)$ *with* s *singletons such that* $|S_j| = 1$ *or* 2 *and such that the associated pair partition is negligible. Then*

$$\lim_{N\to\infty} N^{-k/2} \sum_{(S_1,\cdots,S_p)\in\tilde{\mathcal{P}}_{k,1,2,s}} \sum_{\alpha\in[S_1,\cdots,S_p]} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}) = 0. \tag{21}$$

Proof. Iterating (17), we see that the sum (21) is majorized by

$$CN^{-(k+s)/2} \sum_{(S_1,\cdots,S_p)\in\tilde{\mathcal{P}}_{k,1,2,s}} \sum_{\alpha\in[S_1,\cdots,S_p]} |\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1}\cdots\tilde{g}_{\beta_{k-s}})|, \tag{22}$$

where $(\beta_1,\cdots,\beta_{k-s})$ is obtained from $(\alpha_1,\cdots,\alpha_k)$ by removing the singletons. Since the pair partition associated to $(S_1,\cdots,S_p)$ is negligible, and (20)

$$|\varphi_\gamma(\tilde{g}_{\beta_1}\cdots\tilde{g}_{\beta_{k-s}})|\le c\cdot\frac{\lambda}{\sqrt{N}}$$

so the sum (22) is majorized by a constant times

$$cN^{-(k+s)/2}|\tilde{\mathcal{P}}_{k,1,2,s}|\cdot\frac{\lambda}{\sqrt{N}}\cdot N^p. \tag{23}$$

Since $p=(k+s)/2$ by Lemma13, (23) is dominated by $c/\sqrt{N}\to 0$. ■

Summing up, we come to

THEOREM 16. *Keeping the notations in Definition 5, suppose that the states φ_γ satisfy conditions (17) and (20) for $\gamma\in[0,\bar{\gamma}]$, $\bar{\gamma}>0$. Then it holds that*

$$\lim_{N\to\infty}\varphi_{\lambda/\sqrt{N}}\left(\frac{S_N^{(\epsilon_1)}}{\sqrt{N}}\cdots\frac{S_N^{(\epsilon_k)}}{\sqrt{N}}\right)=\lim_{N\to\infty}N^{-k/2}\sum_{1\le s\le k}\sum_{\alpha}{}'\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}), \tag{24}$$

where $\sum'_\alpha$ means that α runs over the non-negligible pair partitons with s singletons.

Remark. Condition (17) is easily verified for the Haagerup states. In that case the negligible partitions are nothing but the crossing ones. Another examples shall be considered elsewhere.

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